

**LARGE NON-PLANAR GRAPHS AND  
AN APPLICATION TO CROSSING-CRITICAL GRAPHS**

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**ABSTRACT**

We prove that, for every positive integer  $k$ , there is an integer  $N$  such that every 4-connected non-planar graph with at least  $N$  vertices has a minor isomorphic to  $K_{4,k}$ , the graph obtained from a cycle of length  $2k + 1$  by adding an edge joining every pair of vertices at distance exactly  $k$ , or the graph obtained from a cycle of length  $k$  by adding two vertices adjacent to each other and to every vertex on the cycle. We also prove a version of this for subdivisions rather than minors, and relax the connectivity to allow 3-cuts with one side planar and of bounded size. We deduce that for every integer  $k$  there are only finitely many 3-connected 2-crossing-critical graphs with no subdivision isomorphic to the graph obtained from a cycle of length  $2k$  by joining all pairs of diagonally opposite vertices.

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## 1. INTRODUCTION

In this paper *graphs* are finite and may have loops or multiple edges. A graph is a *subdivision* of another if the first can be obtained from the second by replacing each edge by a non-zero length path with the same ends. Our first theorem follows the pattern of the following results. The first two are easy.

(1.1) *For every positive integer  $k$ , there is an integer  $N$  such that every connected graph with at least  $N$  vertices has either a path on  $k$  vertices, or a vertex with at least  $k$  distinct neighbors.*

(1.2) *For every positive integer  $k$ , there is an integer  $N$  such that every 2-connected graph with at least  $N$  vertices has either a cycle of length at least  $k$ , or a subgraph isomorphic to a subdivision of  $K_{2,k}$ .*

These two results were generalized to 3- and 4-connected graphs in [4]. To state the theorems we need to define a few families of graphs. Let  $k \geq 3$  be an integer. The  *$k$ -spoke wheel*, denoted by  $W_k$ , has vertices  $v_0, v_1, \dots, v_k$ , where  $v_1, v_2, \dots, v_k$  form a cycle, and  $v_0$  is adjacent to all of  $v_1, v_2, \dots, v_k$ . The  *$2k$ -spoke alternating double wheel*, denoted by  $A_k$ , has vertices  $v_0, v'_0, v_1, v_2, \dots, v_{2k}$ , where  $v_1, v_2, \dots, v_{2k}$  form a cycle in this order,  $v_0$  is adjacent to  $v_1, v_3, \dots, v_{2k-1}$ , and  $v'_0$  is adjacent to  $v_2, v_4, \dots, v_{2k}$ . The vertices  $v_0$  and  $v'_0$  will be called the *hubs* of  $A_k$ . The  *$k$ -rung ladder*, denoted by  $L_k$ , has vertices  $v_1, v_2, \dots, v_k, u_1, u_2, \dots, u_k$ , where  $v_1, v_2, \dots, v_k$  and  $u_1, u_2, \dots, u_k$  form paths in the order listed, and  $v_i$  is adjacent to  $u_i$  for  $i = 1, 2, \dots, k$ . The graph  $W'_k$  is obtained from  $L_k$  by adding an edge between  $v_1$  and  $v_k$ , and contracting the edges joining  $u_1$  to  $v_1$  and  $u_k$  to  $v_k$ . The graph  $O_k$ , called the  *$k$ -rung circular ladder*, is obtained from  $L_k$  by adding edges between  $v_1$  and  $v_k$  and between  $u_1$  and  $u_k$ ; and the  *$k$ -rung Möbius ladder*, denoted by  $M_k$ , is obtained from  $L_k$  by adding edges between  $v_1$  and  $u_k$  and between  $u_1$  and  $v_k$ . The graph  $K'_{4,k}$  is obtained from  $K_{4,k}$  by splitting each of the  $k$  vertices of degree four in the same way. More precisely, it has vertices  $x, y, x', y', v_1, v_2, \dots, v_k, v'_1, v'_2, \dots, v'_k$ , where  $v_i$  is adjacent to  $v'_i, x$ , and  $y$ , and  $v'_i$  is adjacent to  $v_i, x'$ , and  $y'$  for  $i = 1, 2, \dots, k$ . We remark that  $W_k, W'_k$ , and  $K_{3,k}$  are 3-connected. The following is proved in [4].

(1.3) *For every integer  $k \geq 3$ , there is an integer  $N$  such that every 3-connected graph with at least  $N$  vertices has a subgraph isomorphic to a subdivision of one of  $W_k$ ,  $W'_k$ , and  $K_{3,k}$ .*

For the second result we need a couple more definitions. A *separation* of a graph is a pair  $(A, B)$  of subsets of  $V(G)$  such that  $A \cup B = V(G)$ , and there is no edge between  $A - B$  and  $B - A$ . It is *nontrivial* if  $A - B \neq \emptyset \neq B - A$ . The *order* of  $(A, B)$  is  $|A \cap B|$ . A graph  $G$  is said to be *almost 4-connected* if it is 3-connected and, for every separation  $(A, B)$  of  $G$  of order three, one of  $A - B$ ,  $B - A$  contains at most one vertex. (We remark that this is called “internally 4-connected” in [4], but that term usually has a different meaning.) Clearly every 4-connected graph is almost 4-connected, and if  $k \geq 4$ , then  $A_k$ ,  $O_k$ ,  $M_k$ ,  $K_{4,k}$ , and  $K'_{4,k}$  are almost 4-connected. The following is the second result from [4].

(1.4) *For every integer  $k \geq 4$ , there is an integer  $N$  such that every almost 4-connected graph with at least  $N$  vertices contains a subgraph isomorphic to a subdivision of one of  $A_k$ ,  $O_k$ ,  $M_k$ ,  $K_{4,k}$ , and  $K'_{4,k}$ .*

Our first objective is to prove a version of (1.4) for non-planar graphs, as follows. We define  $B_k$  to be the graph obtained from  $A_k$  by adding an edge joining its hubs.

(1.5) *For every integer  $k \geq 4$ , there is an integer  $N$  such that every almost 4-connected non-planar graph with at least  $N$  vertices has a subgraph isomorphic to a subdivision of one of  $B_k$ ,  $M_k$ ,  $K_{4,k}$ , and  $K'_{4,k}$ .*

A graph is a *minor* of another if the first can be obtained from a subgraph of the second by contracting edges. For the minor containment (1.5) has the following corollary, which was stated for 4-connected graphs in the abstract.

(1.6) *For every integer  $k \geq 4$ , there is an integer  $N$  such that every almost 4-connected non-planar graph with at least  $N$  vertices has a minor isomorphic to  $K_{4,k}$ , or the graph obtained from a cycle of length  $2k + 1$  by adding an edge joining every pair of vertices at distance exactly  $k$ , or the graph obtained from a cycle of length  $k$  by adding two vertices adjacent to each other and to every vertex on the cycle.*

*Proof.* This follows immediately from (1.5), because  $K_{4,k}$  is a minor of  $K'_{4,k}$ ; the second outcome graph is a minor of  $M_{2k+1}$ ; and the third outcome graph is a minor of  $B_{2k}$ .  $\square$

In fact, in (3.4) we prove a stronger result than (1.5). We relax the connectivity requirement on  $G$  to allow separations of order three as long as one side of the separation is planar and has bounded size.

We apply the stronger form of (1.5) to deduce a theorem about 2-crossing-critical graphs. Traditionally, a graph  $G$  is called *2-crossing-critical* if it cannot be drawn in the plane with at most one crossing, but  $G \setminus e$  can be so drawn for every edge  $e \in E(G)$ . (We use  $\setminus$  for deletion. In *drawings* of graphs edges are permitted to cross, whereas in *embeddings* they are not.) But then every graph obtained from a 2-crossing-critical graph by subdividing an edge is again 2-crossing-critical, and (iv) below suggests another simple operation that can be used to generate arbitrarily large 2-crossing-critical graphs. To avoid these easily understood constructions we define a graph  $G$  to be *X-minimal* if

- (i)  $G$  has crossing number at least two,
- (ii)  $G \setminus e$  has crossing number at most one for every edge  $e \in E(G)$ ,
- (iii)  $G$  has no vertices of degree two, and
- (iv)  $G$  does not have a vertex of degree four incident with two pairs of parallel edges.

If  $v$  is a vertex of degree two in a graph  $G$ , and  $G'$  is obtained from  $G$  by contracting one of the edges incident with  $v$ , then  $G$  satisfies (i) if and only if  $G'$  satisfies (i), and the same holds for condition (ii). Similarly, if  $u \in V(G)$  has degree four and is incident with two pairs of parallel edges, and if  $G''$  is obtained from  $G \setminus u$  by adding a pair of parallel edges joining the two neighbors of  $u$ , then the same conclusion holds for  $G$  and  $G''$ . Thus the notion of X-minimality provides a reasonable concept of being “minimal with crossing number at least two”. Our second result then states the following.

**(1.7)** *For every integer  $k$  there exists an integer  $N$  such that every X-minimal graph on at least  $N$  vertices has a subgraph isomorphic to a subdivision of  $M_k$ .*

This is of interest, because of a belief by some experts on crossing numbers that X-minimal graphs with an  $M_7$  subdivision can be completely described. There are infinitely

many of them, but they all seem to fall within a well-described infinite family. The sequel to [1] promises to prove that. Another proof of (1.7) appears in [1].

To prove (1.7) we need the following lemma, which may be of independent interest.

**(1.8)** *Let  $G$  be an  $X$ -minimal graph on at least 17 vertices. Then for every separation  $(A, B)$  of  $G$  of order at most three, one of  $G|A$ ,  $G|B$  has at most 129 vertices and can be embedded in a disk with  $A \cap B$  embedded on the boundary of the disk.*

The bound of 129 is far from best possible, and we make no attempt to optimize it.

The paper is organized as follows. In Section 2 we state two lemmas from other papers that will be used later. In Section 3 we prove (1.5), and in Section 4 we prove a lemma about planar graphs that we use in the final Section 5, where we first prove (1.8) and then (1.7).

The ideas of our paper were initially developed in November 1998 and written in manuscript form [2]. In October 2009 the authors of [1] kindly informed us of their work, and that prompted us to revise [2], resulting in the present article.

## 2. PLANAR SUBGRAPHS OF NON-PLANAR GRAPHS

We formalize the concept of a subdivision as follows. Let  $G, H$  be graphs. A mapping  $\eta$  with domain  $V(G) \cup E(G)$  is called a *homeomorphic embedding* of  $G$  into  $H$  if for every two vertices  $v, v'$  and every two edges  $e, e'$  of  $G$

- (i)  $\eta(v)$  is a vertex of  $H$ , and if  $v, v'$  are distinct then  $\eta(v), \eta(v')$  are distinct,
- (ii) if  $e$  has ends  $v, v'$ , then  $\eta(e)$  is a path of  $H$  with ends  $\eta(v), \eta(v')$ , and otherwise disjoint from  $\eta(V(G))$ , and
- (iii) if  $e, e'$  are distinct, then  $\eta(e)$  and  $\eta(e')$  are edge-disjoint, and if they have a vertex in common, then this vertex is an end of both.

We shall denote the fact that  $\eta$  is a homeomorphic embedding of  $G$  into  $H$  by writing  $\eta : G \hookrightarrow H$ . If  $K$  is a subgraph of  $G$  we denote by  $\eta(K)$  the subgraph of  $H$  consisting of all vertices  $\eta(v)$ , where  $v \in V(K)$ , and all vertices and edges that belong to  $\eta(e)$  for some  $e \in E(K)$ . It is easy to see that  $H$  has a subgraph isomorphic to a subdivision of  $G$  if and only if there is a homeomorphic embedding  $G \hookrightarrow H$ . The reader is advised to notice that

$V(\eta(K))$  and  $\eta(V(K))$  mean different sets. The first is the vertex-set of the graph  $\eta(K)$ , whereas the second is the image of the vertex-set of  $K$  under the mapping  $\eta$ . An  $\eta$ -path in  $H$  is a path in  $H$  with both ends in  $\eta(G)$  and otherwise disjoint from it.

A cycle  $C$  in a graph  $G$  is called *peripheral* if it is induced and  $G \setminus V(C)$  is connected. Let  $\eta : G \hookrightarrow H$ , let  $C$  be a peripheral cycle in  $G$ , and let  $P_1$  and  $P_2$  be two disjoint  $\eta$ -paths with ends  $u_1, v_1$  and  $u_2, v_2$ , respectively, such that  $u_1, u_2, v_1, v_2$  belong to  $V(\eta(C))$  and occur on  $\eta(C)$  in the order listed. In those circumstances we say that the pair  $P_1, P_2$  is an  $\eta$ -cross. We also say that it is an  $\eta$ -cross in  $C$ . We say that  $u_1, v_1, u_2, v_2$  are the *feet* of the cross. We say that the cross is *free* if

- (F1) for  $i = 1, 2$  there is no  $e \in E(G)$  such that  $P_i$  has both ends in  $V(\eta(e))$ , and
- (F2) whenever  $e_1, e_2 \in E(G)$  are such that all the feet of the cross belong to  $V(\eta(e_1)) \cup V(\eta(e_2))$ , then  $e_1$  and  $e_2$  have no end in common.

The following is shown in [6].

**(2.1)** *Let  $G$  be an almost 4-connected planar graph on at least seven vertices, let  $H$  be a non-planar graph, and let  $\eta : G \hookrightarrow H$  be a homeomorphic embedding. Then there exists a homeomorphic embedding  $\eta' : G \hookrightarrow H$  such that  $\eta(v) = \eta'(v)$  for every vertex  $v \in V(G)$  of degree at least four and one of the following conditions holds:*

- (i) *there exists an  $\eta'$ -path in  $H$  such that both of its ends belong to  $V(\eta'(C))$  for no peripheral cycle  $C$  in  $G$ ,*
- (ii) *there exists a free  $\eta'$ -cross, or*
- (iii) *there exists a separation  $(X, Y)$  of  $H$  of order at most three such that  $|\eta'(V(G)) \cap X - Y| \leq 1$  and  $H|X$  does not have an embedding in a disk with  $X \cap Y$  embedded on the boundary of the disk.*

If  $\eta$  is a homeomorphic embedding of  $G$  into  $H$ , an  $\eta$ -bridge is a connected subgraph  $B$  of  $H$  with  $E(B) \cap E(\eta(G)) = \emptyset$ , such that either

- (i)  $|E(B)| = 1$ ,  $E(B) = \{e\}$  say, and both ends of  $e$  are in  $V(\eta(G))$ , or
- (ii) for some component  $C$  of  $H \setminus V(\eta(G))$ ,  $E(B)$  consists of all edges of  $H$  with at least one end in  $V(C)$ .

It follows that every edge of  $H$  not in  $\eta(G)$  belongs to a unique  $\eta$ -bridge. We say that a vertex  $v$  of  $H$  is an *attachment* of an  $\eta$ -bridge  $B$  if  $v \in V(\eta(G)) \cap V(B)$ .

Let  $\eta$  be a homeomorphic embedding of  $G$  into  $H$ . We say that an  $\eta$ -bridge  $B$  is *unstable* if there exists an edge  $e \in E(G)$  such that  $V(B) \cap V(\eta(G)) \subseteq V(\eta(e))$ , and otherwise we say that it is *stable*. The following result is probably due to Tutte. A proof may be found in [3, Lemma 6.2.1] or [6] or elsewhere.

**(2.2)** *Let  $G$  be a graph, let  $H$  be a simple 3-connected graph, and let  $\eta : G \hookrightarrow H$  be a homeomorphic embedding. Then there exists a homeomorphic embedding  $\eta' : G \hookrightarrow H$  such that every  $\eta'$ -bridge is stable and  $\eta(v) = \eta'(v)$  for every vertex  $v \in V(G)$  of degree at least three.*

### 3. LARGE NON-PLANAR GRAPHS

We need the following minor strengthening of (1.4).

**(3.1)** *For every two integers  $k, t \geq 4$  there is an integer  $N$  such that every 3-connected graph with at least  $N$  vertices either contains a subgraph isomorphic to a subdivision of one of  $A_k$ ,  $O_k$ ,  $M_k$ ,  $K_{4,k}$ , and  $K'_{4,k}$ , or it has a separation  $(A, B)$  of order at most three such that  $|A| \geq t$  and  $|B| \geq t$ .*

*Proof.* For  $t = 5$  this is (1.4). For  $t > 5$  the result follows by making obvious modifications to the proof of (1.4) in [4]. □

**(3.2)** *Let  $k \geq 4$  be an integer, let  $H$  be a non-planar graph, and let  $\eta : A_{2k+1} \hookrightarrow H$  be a homeomorphic embedding. Then one of the following holds.*

- (i) *There exist a homeomorphic embedding  $\eta' : A_k \hookrightarrow H$  and an  $\eta'$ -path  $P$  in  $H$  such that  $\eta'$  maps the hubs of  $A_k$  to the same pair of vertices  $\eta$  maps the hubs of  $A_{2k+1}$  to, and the ends of  $P$  are the images of the hubs of  $A_k$  under  $\eta'$ .*
- (ii) *There exist a homeomorphic embedding  $\eta' : A_{2k+1} \rightarrow H$  and a separation  $(A, B)$  of  $H$  of order at most three such that  $|\eta'(V(A_{2k+1})) \cap A - B| \leq 1$  and  $H|A$  cannot be embedded in a disk with  $A \cap B$  embedded in the boundary of the disk.*

*Proof.* By (2.1) we may assume (by replacing  $\eta$  by a different homeomorphic embedding that maps the hubs of  $A_{2k+1}$  to the same pair of vertices of  $H$  as  $\eta$ ) that  $\eta$  satisfies (i), (ii), or (iii) of (2.1). If it satisfies (iii), then the result holds, and so we may assume that  $\eta$  satisfies (2.1)(i) or (2.1)(ii).

Assume first that  $\eta$  satisfies (2.1)(i), and let  $P$  be the corresponding  $\eta$ -path. Let  $v_0, v'_0, v_1, v_2, \dots, v_{4k+2}$  be as in the definition of  $A_{2k+1}$ . If  $P$  has one end in  $V(\eta(v_0v_i)) - \{\eta(v_i)\}$  and the other in  $V(\eta(v'_0v_j)) - \{\eta(v_j)\}$  for some  $i$  and  $j$ , then  $A_{2k+1} \setminus \{v_0v_i, v'_0v_j\}$  has a subgraph  $A$  that is isomorphic to a subdivision of  $A_{2k-1}$ . Let  $\eta'$  be the restriction of  $\eta$  to  $A$  and let  $P'$  be the  $\eta(v_0)\eta(v'_0)$ -path in the union of  $P$ ,  $\eta(v_0v_i)$ , and  $\eta(v'_0v_j)$ . Then  $\eta'$  and  $P'$  satisfy (i).

Thus we may assume by symmetry that both ends of  $P$  are in  $V(\eta(A_{2k+1} \setminus v_0)) - \{\eta(v'_0)\}$ . In fact, we may further assume by symmetry that both ends of  $P$  are in  $V(\eta(A_{2k+1} \setminus \{v_0, v_1, v_2, \dots, v_{2k}\})) - \{\eta(v'_0)\}$ . Since  $P \cup \eta(A_{2k+1})$  is non-planar, there exist  $i, j \in \{2k+1, 2k+2, \dots, 4k+2\}$  with  $|i-j| = 1$  such that  $P$  is vertex-disjoint from  $\eta(Q)$ , where  $Q$  is the path with vertex-set  $\{v_0, v_i, v_j, v'_0\}$ . Let  $\eta'(x) = \eta(x)$  for all vertices and edges  $x$  of  $A_{2k+1} \setminus \{v_{2k+1}, v_{2k+2}, \dots, v_{4k+2}\}$ . We define  $\eta'(v_1v_{2k})$  to be the path in  $H$  with ends  $\eta(v_1)$  and  $\eta(v_{2k})$  consisting of  $P$  and two subpaths of  $\eta(G) \setminus \{\eta(v_0), \eta(v'_0), \eta(v_2), \eta(v_i)\}$ . Then  $\eta' : A_k \hookrightarrow H$  and  $P' = \eta(Q)$  satisfy (i).

The argument is similar when  $\eta$  satisfies (2.1)(ii). □

**(3.3)** *Let  $k \geq 1$  be an integer, and let  $H$  be a non-planar graph such that there exists a homeomorphic embedding  $\eta : O_{4k} \hookrightarrow H$ . Then either  $H$  has a subgraph isomorphic to a subdivision of  $M_k$ , or there exist a homeomorphic embedding  $\eta' : O_{4k} \hookrightarrow H$  and a separation  $(A, B)$  of  $H$  of order at most three such that  $|\eta'(V(O_{4k})) \cap A - B| \leq 1$  and  $H|A$  cannot be embedded in a disk with  $A \cap B$  embedded in the boundary of the disk.*

*Proof.* The proof is similar to that of (3.2). We omit the details. □

Let us recall that  $B_k$  is the graph obtained from  $A_k$  by adding an edge joining its hubs. A graph  $G$  is *t-shallow* if for every separation  $(A, B)$  of order at most three, one of



$G|A, G|B$  has fewer than  $t$  vertices and can be embedded in a disk with  $A \cap B$  embedded on the boundary of the disk. The following is the main result of this section. It implies (1.5), because every almost 4-connected graph is 5-shallow.

**(3.4)** *For every two integers  $k, t \geq 4$  there is an integer  $N$  such that every 3-connected  $t$ -shallow non-planar graph with at least  $N$  vertices contains a subgraph isomorphic to a subdivision of one of  $B_k, M_k, K_{4,k}$ , and  $K'_{4,k}$ .*

*Proof.* Let  $k, t$  be given. By replacing  $k$  by a larger integer we may assume that  $8k \geq t + 1$ . Let  $N$  be the integer that satisfies (3.1) with  $k$  replaced by  $4k$ . We claim that  $N$  satisfies the conclusion of (3.4). To see this let  $G$  be a 3-connected  $t$ -shallow non-planar graph on at least  $N$  vertices. By (3.1)  $G$  has a subgraph isomorphic to a subdivision of one of  $A_{4k}, O_{4k}, M_{4k}, K_{4,4k}$ , and  $K'_{4,4k}$ . If  $G$  has a subgraph isomorphic to a subdivision of  $M_{4k}, K_{4,4k}$ , or  $K'_{4,4k}$ , then the result holds.

Assume now that there exists a homeomorphic embedding  $\eta : A_{4k} \hookrightarrow G$ . By (3.2) either  $G$  has a subgraph isomorphic to a subdivision of  $B_k$ , or there exists a separation  $(A, B)$  as in (3.2)(ii). In the former case the theorem holds, and so we may assume the latter. Since  $G$  is  $t$ -shallow we see that  $|B| < t$ . However, all but possibly one vertex of  $\eta(V(A_{4k}))$  belong to  $B$ , contrary to  $8k \geq t + 1$ . The argument when there exists a homeomorphic embedding  $\eta : O_{4k} \hookrightarrow G$  is similar, using (3.3) instead.  $\square$

#### 4. A LEMMA ABOUT PLANAR GRAPHS

The objective of this section is to prove (4.6). Let  $G$  be a *plane* graph; that is, a graph embedded in the plane. Then every cycle  $C$  bounds a disk in the plane, and we define  $\text{ins}(C)$  to be the set of edges of  $G$  embedded in the open disk bounded by  $C$ . (By definition, an edge of an embedding or drawing does not include its ends.) The following will be a hypothesis common to several lemmas, and so we give it a name in order to avoid repetition.

**(4.1) Hypothesis.** Let  $G$  be a loopless plane graph embedded in the closed unit disk  $\Delta$ , let  $x_1, x_2, x_3$  be distinct vertices of  $G$ , and let them be the only vertices of  $G$  embedded in the boundary of  $\Delta$ . Assume that there is no separation  $(A, B)$  of order at most two with  $x_1, x_2, x_3 \in A$  and  $B - A \neq \emptyset$ .

The last assumption of (4.1) will be referred to as the *internal 3-connectivity* of  $G$ .

Assume (4.1), let  $C$  be a cycle in  $G$  with  $\{x_1, x_2, x_3\} \not\subseteq V(C)$  and  $\text{ins}(C) \neq \emptyset$ . We say that  $C$  is *robust* if there exists an edge  $f \in \text{ins}(C)$  such that for every  $e \in E(C)$  the graph  $G \setminus \{x_1, x_2, x_3\} \setminus e \setminus f$  has a component containing a neighbor of each of  $x_1, x_2, x_3$ . Let  $Z$  be the set of all vertices  $v \in V(C)$  such that either  $v \in \{x_1, x_2, x_3\}$  or  $v$  is incident with an edge not in  $E(C) \cup \text{ins}(C)$ . We say that  $C$  is *flexible* if  $|Z| \leq 3$  and at least two vertices in  $Z - \{x_1, x_2, x_3\}$  are incident with exactly one edge not in  $E(C) \cup \text{ins}(C)$ . Our objective in this section is to prove that if  $G$  has sufficiently many vertices and satisfies Hypothesis (4.1), then it has a robust cycle or a flexible cycle.

**(4.2)** Assume (4.1). Then every cycle of  $G \setminus \{x_1, x_2, x_3\}$  that does not bound a face is robust.

*Proof.* Let  $C$  be a cycle of  $G \setminus \{x_1, x_2, x_3\}$  that does not bound a face, and let  $f \in \text{ins}(C)$ . By the internal 3-connectivity of  $G$  there exist three internally disjoint paths from  $\{x_1, x_2, x_3\}$  to  $V(C)$ , and hence  $G \setminus \{x_1, x_2, x_3\} \setminus e \setminus f$  has a component containing neighbors of all of  $x_1, x_2, x_3$  for all  $e \in E(C)$ . Thus  $C$  is robust, as desired.  $\square$

Let us recall that a *block* is a graph with no cut-vertices, and a *block of a graph* is a maximal subgraph that is a block. The *block graph* of a graph  $G$  is the graph whose vertices are all the blocks of  $G$  and all the cut vertices of  $G$ , with the obvious incidences. An *end-block* of a graph  $G$  is a block that has degree one in the block graph of  $G$ .

**(4.3)** Assume (4.1), and that  $G$  has no robust cycle. Then every two distinct cycles of  $G \setminus \{x_1, x_2, x_3\}$  are edge-disjoint. Consequently, every block of  $G \setminus \{x_1, x_2, x_3\}$  is a cycle or a complete graph on at most two vertices.

*Proof.* This follows from (4.2), because otherwise some cycle of  $G \setminus \{x_1, x_2, x_3\}$  is not facial.  $\square$

**(4.4)** Assume (4.1), and assume that  $G$  has at least 16 vertices and no robust cycle. Let  $B_1, B_2$  be two distinct end-blocks of  $G \setminus \{x_1, x_2, x_3\}$ . For  $i = 1, 2$  let  $v_i$  be the unique cut vertex of  $G \setminus \{x_1, x_2, x_3\}$  that belongs to  $B_i$ , and let  $N_i \subseteq \{x_1, x_2, x_3\}$  be the set of vertices of  $\{x_1, x_2, x_3\}$  that have a neighbor in  $B_i \setminus v_i$ . Then  $|N_1| = |N_2| = 2$  and  $|N_1 \cap N_2| = 1$ .

*Proof.* We first notice that  $N_1$  and  $N_2$  have at least two elements by the internal 3-connectivity of  $G$ . Thus it suffices to show that  $|N_1 \cap N_2| \leq 1$ . Let us assume for a contradiction that  $x_1, x_2 \in N_1 \cap N_2$ . The fact that  $G$  is embedded in a disk with  $x_1, x_2, x_3$  on the boundary of the disk implies that either  $x_3$  has no neighbor outside  $B_1 \setminus v_1$ , or it has no neighbor outside  $B_2 \setminus v_2$ , and hence from the symmetry we may assume the latter. But  $x_3$  has at least one neighbor in  $B_2 \setminus v_2$  by the internal 3-connectivity of  $G$ . Since  $G$  has at least 16 vertices, it follows from (4.3) that  $G \setminus \{x_1, x_2, x_3\}$  has at least seven vertices with at most two neighbors. Each of those vertices has a neighbor in  $\{x_1, x_2, x_3\}$ , and hence there is an index  $i \in \{1, 2, 3\}$  such that  $x_i$  has at least three neighbors in  $G \setminus \{x_1, x_2, x_3\}$ . Furthermore, if  $B_2$  has a unique edge, then  $i$  and the three neighbors of  $x_i$  can be chosen to be not in  $B_2 \setminus v_2$ . Thus there is a cycle  $C$  of  $G$  containing  $x_i$  but no other  $x_j$  such that  $\text{ins}(C)$  includes an edge  $f$  incident with  $x_i$ ; and if  $B_2$  has a unique edge, then  $C$  does not use that edge. Since  $x_1, x_2$  and  $x_3$  all have a neighbor in  $B_2 \setminus v_2$ , it follows that  $C$  is robust, a contradiction.  $\square$

**(4.5)** Assume (4.1), and assume that  $G$  has at least 16 vertices and no robust cycle. Then the block graph of  $G \setminus \{x_1, x_2, x_3\}$  is a path.

*Proof.* Suppose for a contradiction that the block graph of  $G \setminus \{x_1, x_2, x_3\}$  is not a path. Then  $G \setminus \{x_1, x_2, x_3\}$  has at least three end-blocks, say  $B_1, B_2$ , and  $B_3$ . For  $i = 1, 2, 3$  let  $N_i$  be as in (4.4). By (4.4) we may assume that the blocks  $B_1, B_2, B_3$  are numbered in such a way that  $N_1 = \{x_2, x_3\}$ ,  $N_2 = \{x_1, x_3\}$ , and  $N_3 = \{x_1, x_2\}$ . Let  $C$  be a cycle containing an edge joining  $x_i$  to a vertex of  $N_j$  for all distinct integers  $i, j \in \{1, 2, 3\}$ , such that all other

edges of  $C$  belong to  $B_1 \cup B_2 \cup B_3$ . Let  $T$  be a connected subgraph of  $G \setminus \{x_1, x_2, x_3\}$  such that  $V(T \cap C) = \{u_1, u_2, u_3\}$ , where  $u_i \in V(B_i)$ . Then  $x_1, u_3, x_2, u_1, x_3, u_2$  appear on  $C$  in the order listed. Since  $G$  has at least 16 vertices there exist an edge  $f \in E(G) - E(T) - E(C)$  and index  $i \in \{1, 2, 3\}$  such that  $f \in \text{ins}(C')$ , where  $C'$  is the unique cycle in  $(C \cup T) \setminus u_i$ . It follows that  $C'$  is robust, a contradiction.  $\square$

**(4.6)** *Assume (4.1), and assume that  $G$  has at least 130 vertices. Then  $G$  has a robust cycle or a flexible cycle.*

*Proof.* Assume for a contradiction that  $G$  has neither a robust cycle nor a flexible cycle. Let  $B := G \setminus \{x_1, x_2, x_3\}$ , let  $a_1b_1, a_2b_2, \dots, a_tb_t$  be all the cut edges of  $B$ , and let  $D_0, D_1, \dots, D_t$  be all the components of  $B \setminus \{a_1b_1, a_2b_2, \dots, a_tb_t\}$ . By (4.5) the numbering can be chosen so that  $a_j \in V(D_{j-1})$  and  $b_j \in V(D_j)$  for all  $j = 1, 2, \dots, t$ . By (4.4) we may assume that  $x_1$  and  $x_3$  have a neighbor in  $D_0$ , and that  $x_2$  and  $x_3$  have a neighbor in  $D_t$ .

(1) *For  $i \in \{1, 2, 3\}$  and  $j \in \{0, 1, \dots, t\}$  there are at most two edges with one end  $x_i$  and the other end in  $D_j$ .*

To prove (1) suppose for a contradiction that there are three edges with one end  $x_i$  and the other end in  $D_j$ . Then there exists a cycle  $C$  using two of those edges such that the third edge, say  $f$ , belongs to  $\text{ins}(C)$  and  $C \setminus x_i$  is a subgraph of  $D_j$ . If  $0 < j < t$ , then there exists a path  $P$  in  $D_j \setminus E(C)$  with ends  $b_j$  and  $a_{j+1}$ . By considering the edge  $f$  and path  $P$  (when  $0 < j < t$ ) we deduce that  $C$  is robust, a contradiction. This proves (1).

(2) *For  $j = 0, 1, \dots, t$  the graph  $D_j$  has at most 18 vertices.*

To prove (2) we first notice that the block graph of  $D_j$  is a path by (4.5). Since  $D_j$  is 2-edge-connected, each block of  $D_j$  is a cycle by (4.3). By the internal 3-connectivity of  $G$  no two consecutive blocks of  $D_j$  are both a cycle of length two, unless their shared vertex is adjacent to at least one of  $x_1, x_2, x_3$ . Since every vertex of  $D_j$  except possibly  $b_j$  (if  $j > 0$ ) and  $a_{j+1}$  (if  $j < t$ ) has at least three distinct neighbors by the internal 3-connectivity of  $G$ , the claim follows from (1). This proves (2).

(3) *There is at most one index  $j \in \{1, 2, \dots, t-1\}$  such that the graph  $D_j$  includes a neighbor of  $x_1$ .*

To prove (3) we suppose for a contradiction that there exist two such indices  $j, j'$  with  $0 < j' < j < t$ . Since  $x_1$  has also a neighbor in  $B_0$ , there exists a cycle  $C$  through the vertex  $x_1$  with  $V(C) \subseteq V(D_0 \cup D_1 \cup \dots \cup D_j) \cup \{x_1\}$  and such that some edge  $f$  incident with  $x_1$  belongs to  $\text{ins}(C)$ . Since  $x_2$  and  $x_3$  have a neighbor in  $D_t$ , and  $D_j$  is 2-edge-connected, it follows that  $C$  is robust, a contradiction. This proves (3).

From the symmetry between  $x_1$  and  $x_2$  we deduce

(4) *There is at most one index  $j \in \{1, 2, \dots, t-1\}$  such that the graph  $D_j$  includes a neighbor of  $x_2$ .*

We are now ready to complete the proof of the lemma. Since  $G$  has at least 130 vertices, it follows from (2) that  $t \geq 8$ , and hence by (3) and (4) there exists an integer  $j \in \{1, 2, \dots, t-2\}$  such that both  $D_j$  and  $D_{j+1}$  include no neighbor of  $x_1$  or  $x_2$ . Thus each of them includes a neighbor of  $x_3$  by the internal 3-connectivity of  $G$ , and hence there exists a cycle  $C$  with  $V(C) \subseteq V(D_j \cup D_{j+1}) \cup \{x_3\}$ ,  $x_3, b_j, a_{j+2} \in V(C)$ , and such that  $a_j b_j$  is the only edge of  $G$  incident with  $b_j$  that does not belong to  $E(C) \cup \text{ins}(C)$ , and  $a_{j+2} b_{j+2}$  is the only such edge incident with  $a_{j+2}$ . By considering the set  $Z = \{a_{j+2}, b_j, x_3\}$  we deduce that  $C$  is flexible, as desired.  $\square$

We also need the following mild strengthening of (4.6). If  $C$  is a subgraph of a graph  $G$ , then by a  $C$ -bridge we mean an  $\eta$ -bridge, where  $\eta : C \hookrightarrow G$  is the homeomorphic embedding that maps every vertex and edge of  $C$  onto itself.

**(4.7)** *Assume (4.1), and let  $C$  be a robust or flexible cycle in  $G$  with  $\text{ins}(C)$  maximal. Then for every  $C$ -bridge  $B$  of  $G$  either  $E(B) \subseteq \text{ins}(C)$ , or at least one of  $x_1, x_2, x_3$  belongs to  $V(B) - V(C)$ .*

*Proof.* Assume first that  $C$  is robust, let  $f \in \text{ins}(C)$  be as in the definition of robust, and suppose for a contradiction that  $B$  is a  $C$ -bridge that satisfies neither conclusion of the lemma. By the internal 3-connectivity of  $G$  the bridge  $B$  includes a path  $P$  of

$G \setminus \{x_1, x_2, x_3\}$  with both ends on  $C$ , and otherwise disjoint from it. The graph  $C \cup P$  includes a cycle  $C' \neq C$  with  $\text{ins}(C)$  properly contained in  $\text{ins}(C')$ . Since every edge of  $P$  belongs to a cycle of  $G \setminus f$  it follows that  $C'$  is robust, contrary to the maximality of  $C$ .

The argument when  $C$  is flexible is similar. In that case the set  $Z$  from the definition of flexible is the same for  $C$  and  $C'$ . □

## 5. LARGE GRAPHS WITH CROSSING NUMBER AT LEAST TWO

Recall that a graph  $G$  is *X-minimal* if

- (i)  $G$  has crossing number at least two,
- (ii)  $G \setminus e$  has crossing number at most one for every edge  $e \in E(G)$ ,
- (iii)  $G$  has no vertices of degree two, and
- (iv)  $G$  does not have a vertex of degree four incident with two pairs of parallel edges.

**(5.1)** *Every X-minimal graph on at least 17 vertices is 3-connected.*

*Proof.* Let  $G$  be an X-minimal graph on at least 17 vertices, and suppose for a contradiction that it is not 3-connected. Thus it has a nontrivial separation  $(A, B)$  of order at most two. We may assume that  $(A, B)$  has the minimum order among all nontrivial separations of  $G$ .

Assume first that the order of  $(A, B)$  is at most one. Both  $G|A$  and  $G|B$  have crossing number at most one by the X-minimality of  $G$ . They are both non-planar, for otherwise  $G$  itself would have crossing number at most one. Thus both  $G|A$  and  $G|B$  have subgraphs isomorphic to subdivisions of  $K_5$  or  $K_{3,3}$  by Kuratowski's theorem. Now the X-minimality of  $G$  implies that  $G|A$  and  $G|B$  have at most seven vertices, contrary to the fact that  $G$  has at least 17 vertices.

We may therefore assume that  $G$  is 2-connected and that the order of  $(A, B)$  is two. Let  $A \cap B = \{u, v\}$ . Let  $G_1$  be the graph obtained from  $G|A$  as follows. If  $G|B$  has two edge-disjoint paths with ends  $u$  and  $v$ , then  $G_1$  is obtained from  $G|A$  by adding two edges with ends  $u$  and  $v$ ; otherwise  $G_1$  is obtained from  $G|A$  by adding one edge with ends  $u$  and  $v$ . We define  $G_2$  analogously (with the roles of  $A$  and  $B$  interchanged).

(1) *The graphs  $G_1$  and  $G_2$  have crossing number at most one.*

To prove (1) it suffices to argue for  $G_1$ . Assume first that  $G|B$  does not have two edge-disjoint paths with ends  $u$  and  $v$ . Since  $G|B$  has a path with ends  $u$  and  $v$  by the 2-connectivity of  $G$ , we deduce that a subdivision of  $G_1$  is isomorphic to a subgraph of  $G$ , and that the containment is proper. Thus  $G_1$  has crossing number at most one by the X-minimality of  $G$ . We may therefore assume that  $G|B$  has two edge-disjoint paths  $P_1$  and  $P_2$  with ends  $u$  and  $v$ . Then by choosing the paths with  $P_1 \cup P_2$  minimum it can be arranged that both  $P_1$  and  $P_2$  pass through the vertices of  $V(P_1) \cap V(P_2)$  in the same order. The graph  $(G|A) \cup P_1 \cup P_2$  is a proper subgraph of  $G$  by the X-minimality of  $G$ , and hence has crossing number at most one. It follows that  $G_1$  has crossing number at most one. This proves (1).

(2) *The graphs  $G_1$  and  $G_2$  are non-planar.*

To prove (2) it again suffices to argue for  $G_1$ . Suppose for a contradiction that  $G_1$  is planar. By (1) there exists a planar drawing of  $G_2$  with at most one crossing. If none of the edges of  $E(G_2) - E(G|B)$  is involved in the crossing, then this drawing and a planar embedding of  $G_1$  can be combined to produce a planar drawing of  $G$  with at most one crossing. Thus we may assume that an edge of  $E(G_2) - E(G|B)$  is crossed by another. Therefore we may assume that  $E(G_2) - E(G|B)$  consists of a unique edge, say  $e$ , and hence, by construction,  $G_1$  does not have two edge-disjoint paths with ends  $u$  and  $v$ . By Menger's theorem  $G_1$  has an edge  $f$  such that  $G_1 \setminus f$  has no path between  $u$  and  $v$ . Using the drawings of  $G_1$  and  $G_2$  it is now possible to obtain a drawing of  $G$ , where  $e$  and  $f$  are the only two edges that cross, contrary to the fact that  $G$  has crossing number at least two. This proves (2).

From (2) and Kuratowski's theorem it follows that for  $i = 1, 2$  the graph  $G_i$  has a subgraph  $H_i$  isomorphic to a subdivision of  $K_5$  or  $K_{3,3}$ . But  $H_1 \cup H_2$  has crossing number at least two, and hence the X-minimality of  $G$  implies that both  $G_1$  and  $G_2$  have at most eight vertices, contrary to the fact that  $G$  has at least 17 vertices. This proves that  $G$  is 3-connected.  $\square$

(5.2) Let  $G$  be a graph, let  $C$  be a cycle in  $G$ , and let  $B_0, B_1, \dots, B_k$  be the  $C$ -bridges of  $G$  such that the graph  $C \cup B_1 \cup B_2 \cup \dots \cup B_k$  has a planar drawing with no crossings in which  $C$  bounds a face. Let  $H$  denote the graph  $C \cup B_0$ , and let  $f \in E(B_1)$ . Assume further that either  $G \setminus e \setminus f$  is non-planar for every  $e \in E(C)$ , or that the  $C$ -bridge  $B_0$  has exactly three attachments, two of which have degree three in  $H$ . If  $G \setminus f$  has crossing number at most one, then so does  $G$ .

*Proof.* Let  $\Gamma$  be a drawing of  $G \setminus f$  with at most one crossing. Our first objective is to modify  $\Gamma$  to produce a drawing of  $H$  with at most one crossing such that no edge of  $C$  is crossed by another edge. If no edge of  $C$  is crossed by another edge in the drawing  $\Gamma$ , then its restriction to  $H$  is as desired. Thus we may assume that an edge  $e \in E(C)$  is crossed by another edge  $e'$  in  $\Gamma$ . It follows that  $G \setminus e \setminus f$  is planar, and hence, by hypothesis, the  $C$ -bridge  $B_0$  has exactly three attachments, say  $v_1, v_2, v_3$ , such that  $v_1$  and  $v_2$  have degree three in  $H$ . If  $e' \notin E(B_0)$ , then it is easy to convert  $\Gamma$  to a desired drawing of  $H$ . Thus we may assume that  $e' \in E(B_0)$ . It follows that  $B_0 \setminus e'$  has two components, say  $J_1$  and  $J_2$ , such that  $J_1$  is drawn in the closed disk bounded by  $C$  and  $J_2$  is drawn in the closure of the other face of  $C$ . Using the fact that  $v_1$  and  $v_2$  have degree three in  $H$  it is now easy to draw  $J_2$  in the closed disk bounded by  $C$  so as to obtain a desired drawing of  $H$ . This proves our claim that  $H$  has a drawing with at most one crossing such that no edge of  $C$  is crossed by another edge in that drawing. Thus  $C$  bounds a face. By hypothesis it is possible to draw  $B_1 \cup B_2 \cup \dots \cup B_k$  without crossings in that face, showing that  $G$  has crossing number at most one, as desired.  $\square$

Let  $G$  be a graph, let  $u, u_1, u_2, u_3$  be distinct vertices of  $G$ , and let  $Q_1, Q_2, Q_3$  be three paths in  $G$  such that  $Q_i$  has ends  $u$  and  $u_i$  and such that  $Q_1, Q_2, Q_3$  are disjoint except for  $u$ . We say that  $Q_1 \cup Q_2 \cup Q_3$  is a *triad* in  $G$ , and that the vertices  $u_1, u_2, u_3$  are its *feet*. Let  $G$  be a graph, and let  $P_1, P_2, P_3$  be three pairwise disjoint paths in  $G$ , where  $P_i$  has ends  $u_i$  and  $v_i$ . Let  $T_1$  and  $T_2$  be two triads with feet  $v_1, v_2, v_3$  such that the graphs  $P_1 \cup P_2 \cup P_3, T_1, T_2$  are pairwise disjoint, except for  $v_1, v_2, v_3$ . In those circumstances we say that  $P_1 \cup P_2 \cup P_3 \cup T_1 \cup T_2$  is a *tripod*, and that the vertices  $u_1, u_2, u_3$  are its *feet*. We need the following result of [5].



(5.3) Let  $G$  be a graph, and let  $u_1, u_2, u_3$  be three vertices of  $G$  such that there is no separation  $(A, B)$  of  $G$  of order at most two with  $u_1, u_2, u_3 \in A$  and  $B - A \neq \emptyset$ . If  $G$  has no planar embedding with the vertices  $u_1, u_2, u_3$  incident with the same face, then  $G$  has a tripod with feet  $u_1, u_2, u_3$ .

(5.4) Let  $G$  be an  $X$ -minimal graph on at least 17 vertices, and let  $(A, B)$  be a separation in  $G$  of order three. Then one of  $G|A$ ,  $G|B$  has a planar embedding with the vertices  $A \cap B$  embedded on the boundary of the same face.

*Proof.* Suppose for a contradiction that the conclusion does not hold. By (5.1) the graph  $G$  is 3-connected. By (5.3)  $G|A$  has a tripod  $T_1$  with feet  $A \cap B$ , and  $G|B$  has a tripod  $T_2$  with feet  $A \cap B$ . The graph  $T_1 \cup T_2$  has crossing number at least two, as is easily seen. Thus  $G = T_1 \cup T_2$  by the  $X$ -minimality of  $G$ . Moreover, the  $X$ -minimality of  $G$  implies that  $G$  has at most 10 vertices, a contradiction.  $\square$

We are now ready to prove (1.8), which we restate.

(5.5) Every  $X$ -minimal graph on at least 17 vertices is 130-shallow.

*Proof.* Let  $G$  be an  $X$ -minimal graph on at least 17 vertices, and let  $(A, B)$  be a separation in  $G$  of order at most three with  $A - B \neq \emptyset \neq B - A$ . By (5.1) the separation  $(A, B)$  has order exactly three. By (5.4) we may assume that  $G|B$  is embedded in a disk with the vertices of  $A \cap B$  embedded in the boundary of the disk. It follows that  $G|B$  satisfies (4.1), where  $A \cap B = \{x_1, x_2, x_3\}$ . We may and shall assume for a contradiction that  $|B| \geq 130$ . By (4.6) applied to the graph  $G|B$  we deduce that  $G|B$  has a cycle  $C$  that is robust or flexible. By (4.7) we may choose  $C$  so that exactly one  $C$ -bridge  $B_0$  of  $G$  satisfies  $E(B_0) \not\subseteq \text{ins}(C)$ . We wish to apply (5.2), and so we need to verify the hypotheses. If  $C$  is robust, then let  $f$  be as in the definition of robust; otherwise let  $f \in \text{ins}(C)$  be arbitrary. If  $C$  is flexible, then the bridge  $B_0$  has exactly three attachments, and two of them have degree three in  $C \cup B_0$ . Now let  $C$  be robust, and let  $e \in E(C)$ . We claim that  $G \setminus e \setminus f$  is not planar. To prove this we first notice that  $G|A$  cannot be embedded in a disk with  $A \cap B$  embedded in the boundary of the disk, because  $G|B$  can be so embedded and  $G$  is

not planar. By (5.3) the graph  $G|A$  has a tripod  $T$  with feet  $A \cap B$ . Since  $C$  is robust the graph  $(G|B) \setminus e \setminus f$  has a connected subgraph  $R$  that includes  $A \cap B$ . It follows that  $T \cup R$  is a subdivision of  $K_{3,3}$ , which proves our claim that  $G \setminus e \setminus f$  is not planar. The graph  $G \setminus f$  has crossing number at most one by the X-minimality of  $G$ , and hence by (5.2) the graph  $G$  has crossing number at most one, a contradiction.  $\square$

**(5.6)** *Let  $G$  be the graph obtained from  $A_4$  by subdividing the edges  $v_1v_2$  and  $v_5v_6$ , and joining the new vertices by an edge. Then  $G$  has crossing number at least two.*

*Proof.* This follows from the fact that the new edge is the only edge  $e \in E(G)$  such that  $G \setminus e$  is planar.  $\square$

**(5.7)** *No X-minimal graph has a subgraph isomorphic to a subdivision of  $B_{65}$ .*

*Proof.* Let  $H$  be an X-minimal graph, and suppose for a contradiction that it has a subgraph isomorphic to a subdivision of  $B_{65}$ . Let  $\eta : B_{65} \hookrightarrow H$  be a homeomorphic embedding, and let  $\eta_0$  be the restriction of  $\eta$  to  $A_{65}$ . Let  $e_0$  be the edge of  $B_{65}$  joining the two hubs. Let  $J$  be the union of  $\eta_0(A_{65})$  and all  $\eta_0$ -bridges except the one that includes  $\eta(e_0)$ . We claim that  $J$  is planar. To prove this claim suppose for a contradiction that it is not. From (3.2) applied to  $A_{65}, J$ , and  $\eta_0$  we deduce that (i) or (ii) of (3.2) holds. If (i) holds, then we conclude that the graph obtained from  $B_{32}$  by adding an edge parallel to  $e_0$  is isomorphic to a subdivision of  $H$ . That is a contradiction, because said graph is not X-minimal, as is easily seen. Thus we may assume that (3.2)(ii) holds; that is,  $H$  has a separation  $(A, B)$  as in (3.2)(ii). But  $|B| \geq |V(B_{65})| - 1 \geq 130$ , and  $H|A$  does not have a planar embedding with the vertices in  $A \cap B$  incident with the same face, contrary to (5.5). This proves our claim that  $J$  is planar. Thus we may regard  $J$  as a graph embedded in the sphere.

Let the vertices of  $A_{65}$  be numbered as in the definition of  $A_{65}$ . Assume first that  $\eta(e_0)$  has only one edge. Let  $C_0$  be a cycle in  $J$  with  $v_0 \notin V(C_0)$  such that the open disk bounded by  $C_0$  that includes  $v_0$  is as small as possible. Let  $C'_0$  be defined analogously, with  $v'_0$  replacing  $v_0$ . The cycles  $C_0, C'_0$  are edge-disjoint, for otherwise  $H$  has crossing

number at most one. But now it follows that the graph obtained from  $H$  by deleting an edge of  $\eta(v_0v_1)$  has crossing number at least two, contrary to the X-minimality of  $H$ . This completes the case when  $\eta(e_0)$  has only one edge.

We may therefore assume that  $\eta(e_0)$  has at least one internal vertex. Let us say that an  $\eta$ -bridge of  $H$  is *solid* if either it has at least two edges, or it has a unique edge and that edge is not parallel to an edge of  $\eta(B_{65})$ . By (2.2) we may assume that every solid  $\eta$ -bridge is stable. Let us say that a vertex  $v \in V(\eta_0(A_{65})) - \{\eta_0(v_0), \eta_0(v'_0)\}$  is *exposed* if there exists an  $\eta$ -path between an internal vertex of  $\eta(e_0)$  and  $v$ . It follows from (5.1) that there exists at least one exposed vertex. For an integer  $i \in \{1, 3, \dots, 129\}$  let  $C_i$  denote the cycle of  $A_{65}$  with vertex-set  $\{v_i, v_{i+1}, v_{i+2}, v_{i+3}, v_{i+4}, v_0\}$  (index arithmetic modulo 130), and let  $F_i$  be the set of edges of  $A_{65}$  with at least one end in  $V(C_i)$ . From (5.6) we deduce that there exists an integer  $i$  such that  $\eta(e)$  includes an exposed vertex for no  $e \in F_i$ . Let  $J_0, J_1, \dots, J_k$  be all the  $\eta(C_i)$ -bridges of  $H$ , where  $J_0$  includes  $v'_0$ . Then  $J_0$  includes  $\eta(e_0)$ , and hence  $J_1, J_2, \dots, J_k$  are also  $\eta_0(C_i)$ -bridges of  $J$ . Since every solid  $\eta$ -bridge is stable, it follows that  $J_1, J_2, \dots, J_k$ , when regarded as  $\eta_0(C_i)$ -bridges of  $J$ , are embedded in the closed disk  $\Delta$  bounded by  $\eta_0(C_i)$  that does not include  $v'_0$ ; hence  $\eta_0(C_i) \cup J_1 \cup J_2 \cup \dots \cup J_k$  has a planar embedding in which  $\eta_0(C_i)$  bounds a face. Since in the planar embedding of  $J$  the path  $\eta(v_0v_{i+2})$  is embedded in  $\Delta$  we deduce that  $k \geq 1$ . Thus we may select  $f \in E(J_1)$ . Since there exists an exposed vertex, but none in  $\eta(e)$  for any  $e \in F_i$ , it follows that  $H \setminus e \setminus f$  is non-planar for every edge  $e \in E(C_i)$ . The graph  $H \setminus f$  has crossing number at most one by the X-minimality of  $G$ , contrary to (5.2).  $\square$

We are finally ready to prove (1.7), which we restate.

**(5.8)** *For every integer  $k$  there exists an integer  $N$  such that every X-minimal graph on at least  $N$  vertices has a subgraph isomorphic to a subdivision of  $M_k$ .*

*Proof.* We may assume that  $k \geq 65$ . Let  $N$  be such that (3.4) holds for  $k$  and  $t := 130$ , and let  $G$  be an X-minimal graph on at least  $N$  vertices. By (5.5) the graph  $G$  is 130-shallow. By (3.4) it has a subgraph isomorphic to a subdivision of one of  $B_k$ ,  $M_k$ ,  $K_{4,k}$ , and  $K'_{4,k}$ . But  $G$  clearly has no subgraph isomorphic to a subdivision of  $K_{4,k}$  or  $K'_{4,k}$  (because the

crossing number of these graphs is too large), and it has no subgraph isomorphic to a subdivision of  $B_k$  by (5.7), because  $k \geq 65$ . Thus  $G$  has a subgraph isomorphic to a subdivision of  $M_k$ , as desired.  $\square$

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